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Test Problem Construction for Linear Bilevel Programming Problems

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Abstract. A method of constructing test problems for linear bilevel programming problems is presented. The method selects a vertex of the feasible region, 'far away' from the solution of the relaxed linear programming problem, as the global solution of the bilevel problem. A predetermined number of constraints are systematically selected to be assigned to the lower problem. The proposed method requires only local vertex search and solutions to linear programs.

Keywords: Bilevel programming problem, nonconvex programming, test problems, global optimization.

1. Introduction

The Bilevel programming problem belongs to a class of nonconvex global optimization problems with numerous and diverse applications (see Vicente and Calamai [24] for a recent comprehensive review of the literature). As a result, bilevel programming has taken an important role in the field of global optimization [1], [22], [23], [24]. A number of algorithms has been proposed to solve linear bilevel and to some extent nonlinear bilevel optimization problems [2], [8], [24]. To evaluate these algorithms and measure their efficiency, complexity, and applicability, often one needs to have a variety of test problems with known global solutions. To date, there have been many papers on test generation of other nonconvex optimization problems such as reverse convex programming [14], [16], concave minimization [9], [14], [15], [16], [18], and quadratic programs [18], [19], [20]. No methods have been available for generation of bilevel programming test problems, except for the recent work of Calamie and Vicente [6], [7]. Their method requires a specific construction of a two-variable problem with only one controlling parameter. The

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number of variables is increased by combining a number of two-variable problems to obtain a separable multivariable problem. The separable problem is then disguised by the use of a clever transformation.

The purpose of this paper is to present a technique for generating bilevel programming test problems in R^n that need not be separable in the constraints and do not require any transformations. Our method follows the path of constructing a random polytope in R^n and then selecting a vertex of this polytope that is bilevel optimal. The proposed method requires only the use of linear programming and some local vertex search. The method is easy to implement numerically.

The paper is divided as follows: Section 2 describes the basic properties of linear bilevel programming problems which are essential in our construction procedure. Section 3 describes the method of selecting a first optimal solution candidate, and in section 4 the lower-level constraint selection is presented. In that latter section the global solution is also selected. Section 5 presents the random construction of the constraint set for the upper and the lower-level problems along with the right-hand side vector and the upper and the lower-level objective functions. Finally, in section 6 a numerical example illustrates the method.

2. Properties of Linear Bilevel Programming Problem

A *linear bilevel programming* problem is defined as follows,

$$\begin{aligned}
&\min_{(x,y)\geq 0} \quad c_1^\top x + d_1^\top y \\
&\qquad\quad A_1 x + B_1 y \leq b_1 \\
&\text{where y solves,} \qquad\qquad\qquad\qquad (P)\\
&\qquad\quad \min_{z\geq 0} \quad c_2^\top x + d_2^\top z \\
&\qquad\qquad\quad A_2 x + B_2 z \leq b_2
\end{aligned}$$

where A_1, B_1, A_2, and B_2 are matrices of size $(m_1 \times n_1)$, $(m_1 \times n_2)$, $(m_2 \times n_1)$, $(m_2 \times n_2)$, respectively; vectors $x \in R^{n_1}$, $y \in R^{n_2}$, $b_1 \in R^{m_1}$, and $b_2 \in R^{m_2}$.
The lower problem involves only minimization over y; and therefore, without loss of generality, we subsequently assume $c_2 = 0$.
Let,

$$\psi(x) = \min\{d_2^\top y \mid B_2 y \leq b_2 - A_2 x,\ y \geq 0\}$$

denote the optimal value function of the lower level problem. In addition let,

$$g(x,y) = d_2^\top y - \psi(x),$$

and

$$P = \{(x,y) \mid Ax + By \leq b,\ x \geq 0,\ y \geq 0\},$$

denote a gap function, and a bounded nonempty polyhedron in R^n, for $n = n_1 + n_2$, respectively; where,

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Clearly problem (P) is equivalent to the following *facial reverse convex program,*

$$\begin{aligned} \min\ & c_1^T x + d_1^T y \\ \text{s.t.}\ & (x, y) \in P \qquad\qquad (Q) \\ & g(x, y) \leq 0 \end{aligned}$$

It is well known that an optimal solution of a reverse convex problem is on the intersection of the boundary of the reverse convex constraint, ∂g, and an edge of the polytope P (see Hillestad and Jacobsen [11]). The following proposition [5], [22] will show that in the case of a facially reverse convex constraint problem, an optimal solution is at a vertex of P.

PROPOSITION 1 *If* (Q) *is solvable then an optimal solution is achieved at a vertex of P.*

Proof: If (x, y) solves (Q) then $(x, y) \in P$ and $d_2^T y = \psi(x)$. Define the feasible set $F = \{(x, y) \in P |\ g(x, y) = 0\}$. If $F \neq \phi$, then F is the set of optimal solutions of the following concave minimization problem,

$$\min \{g(x, y) |\ (x, y) \in P\},$$

and thus F is the union of some the faces of P (see Rockafellar[21], theorem 32.1). Since P is compact and the optimal solution of (Q) must minimize the linear function $c_1^T x + d_1^T y$ on F, it follows that at least one solution is achieved at a vertex of P. □

Let $M(x)$, the set of optimal solution of the lower problem, be the follower's *rational reaction* set to a given x. The union of all possible vectors that the upper level may select, $x \in P$, and the corresponding 'rational reaction' by the lower level, $y \in M(x)$, is called the *Inducible Region* (IR), a terminology borrowed from Bard [3]. It has been shown that the inducible region is connected [10](an excellent description of the geometry and properties of bilevel programming is given in Benson [4]).

It is clear from the geometry of the problem and the proof of proposition 1 that the inducible region is on the faces of the polytope P and hence the phrase *facial reverse convex program* for the problem (Q).

The above properties are important components of the test generation technique proposed in this paper.

3. Method of Construction

Given arbitrary A, B, b, c_1, d_1, and $P = \{(x,y) \mid Ax + By \leq b,\ x \geq 0,\ y \geq 0\}$, where P is not empty and bounded, we wish to partition $[A, B] = [A_1, A_2; B_1, B_2]$ in order to create a problem of the form (P) for which we know the optimal solution. Let,

$$(\hat{x}, \hat{y}) \in Argmax\{c_1^\top x + d_1^\top y \mid (x,y) \in P\}.$$

$$(\bar{x}, \bar{y}) \in Argmin\{c_1^\top x + d_1^\top y \mid (x,y) \in P\}.$$

Select $(x', y') \in P$ such that $(x', y') = (\bar{x}, \bar{y}) + \mu \cdot ((\hat{x}, \hat{y}) - (\bar{x}, \bar{y}))$, $\mu \in (0,1)$. Furthermore, let $z' = c_1^\top x' + d_1^\top y'$, and define the half-space H' as,

$$H' = \{(x,y) \mid c_1^\top x + d_1^\top y \geq z'\}.$$

Let,

$$\Omega = P \cap H'$$

Clearly, $(\bar{x}, \bar{y}) \notin \Omega$. Let $\tilde{s} = (\tilde{x}, \tilde{y})$ denote the optimal solution of the following linear programming problem,

$$\tilde{s} = (\tilde{x}, \tilde{y}) = Argmin\{c_1^\top x + d_1^\top y \mid (x,y) \in \Omega\}.$$

Assume that $\tilde{s}$ is a nondegenerate vertex of Ω that lies on an edge of P. Let $s = (x,y)$ be a neighboring vertex of $\tilde{s} = (\tilde{x}, \tilde{y})$ such that $c_1^\top x + d_1^\top y > c_1^\top \tilde{x} + d_1^\top \tilde{y}$. In practice, s can be found by computing the basis vector of the *null space* of the active constraints of P at $\tilde{s}$, or by the use of the optimal Simplex tableau associated with $\tilde{s}$. Next, solve the following problem by fixing the x component of $s = (x,y)$,

$$\begin{aligned} \min \quad & d_2^\top y \\ \text{s.t.} \quad & By \leq b - Ax \\ & y \geq 0, \end{aligned}$$

and denote its optimal solution by y^ℓ. If y^ℓ is identical to y then (x, y^ℓ) or equivalently (x,y) is bilevel feasible (i.e. $s \in \mathrm{IR}$). Hence $s^\circ = (x^\circ, y^\circ)$, the first candidate for the global solution of the generated bilevel programming test problem, is initialized at $s = (x,y)$. On the other hand, if $y^\ell \neq y$ then $(x, y^\ell) \in \partial P$, where ∂P denotes the boundary of P, and select s° by the following *active set strategy*. More specifically, let I^+ denote the set of active constraints of P at the point $s^\ell = (x, y^\ell)$, and I^- denote the set of inactive constraints at the point s^ℓ.
Consequently I^+, I^- can be defined as follows,

$$I^+ = \{i \mid A_i x + B_i y = b_i\}$$

$$I^- = \{i \mid A_i x + B_i y < b_i\},$$

where I^+, and I^- include the non-negativity constraints as well and, therefore, $\{I^+ \cup I^-\}$ enumerates all the constraints of P. Let $s^\circ = (x^\circ, y^\circ)$ denote the optimal solution of the following subproblem,

$$\min_{x,y} \quad c_1^\top x + d_1^\top y$$
$$\text{s.t.}$$
$$A_i x + B_i y = b_i, \quad i \in I^+ \qquad\qquad (\mathrm{SP}(s^\ell))$$
$$A_i x + B_i y \leq b_i, \quad i \in I^-$$

4. Lower-Level Constraints Selection

Thus far, we have designated a vertex, s°, to be a candidate for the global solution of the bilevel programming problem. Next, we will identify certain rows of P that will be designated as the lower level constraint set in order to ensure that s° is not a trivial solution of the bilevel problem.

To this end, define a lower bound $z^\circ = c_1^\top x^\circ + d_1^\top y^\circ + \epsilon$, for $\epsilon > 0$ and small. Let I denote the index of the constraint set for the lower level problem, where initially $I = \phi$. Constraints are appended to the set I according to the following strategy. Select $m_\circ < n$ and let,

$$T_\circ = \{1, \ldots, m_\circ\},$$

denote the index set of the subset of the tight constraints of P at $s^\circ = (x^\circ, y^\circ)$. For each $j \in T_\circ$, let $I^+ = \{j\}$ and $s^j = (x^j, y^j)$ be the solution of the subproblem $(SP(s^\circ))$. Constraint j is appended to the set I if $c_1^\top x^j + d_1^\top y^j > z^\circ$,

$$I = I \cup \{j\}.$$

Assume that we have selected a number of constraints, $I = \{1, \ldots, p\}$, $p < m$, where $m = m_1 + m_2$, and that we are at some vertex $v^k = (x^k, y^k)$ of P. Let $N(v^k)$ denote the set of all the neighboring vertices of v^k, and let,

$$NV(v^k) = \{v \in N(v^k) \mid c_1^\top x + d_1^\top y \;>\; c_1^\top x^k + d_1^\top y^k\}.$$

Select a vertex $v^j \in NV(v^k)$, and let $T(v^j) = \{1, \ldots, r\}$, $r \geq n$, denote the index set of tight constraints of P at v^j. Select T_j, a small arbitrary subset of $T(v^j)$, and for each $i \in T_j$ denote by $s^i = (x^i, y^i)$ the optimal solution of the linear programming subproblem $(SP(v^j))$.
If,

$$c_1^\top x^i + d_1^\top y^i \geq z^\circ,$$

then append the i^{th} constraint to the set I, $I = I \cup \{i\}$. On the other hand, if

$$z' < c_1^\top x^i + d_1^\top y^i < z^\circ,$$

then reset s° to s^i, and $I = I \cup \{i\}$. Let $k = j$ and continue the process until a predetermined number of constraints is systematically selected. We have constructed the following linear bilevel programming problem,

$$\min_{(x,y)\geq 0} \quad c_1^\top x + d_1^\top y$$

$$A_i x + B_i y \leq b_i, \quad i \notin I$$

where y solves,

$$\min_{z\geq 0} \quad d_2^\top z$$

$$A_i x + B_i z \leq b_i \quad i \in I,$$

and by way of construction, the inducible region remains connected and the global solution of the above bilevel programming problem is at vertex $s^\circ = (x^\circ, y^\circ)$.

The theorem below, the proof of which has already been given above, summarizes the main result of this section.

THEOREM 1 *Let a polytope P, a nondegenerate vertex $s^\circ = (x^\circ, y^\circ)$ of P, and the set I be given with the desired properties as described in this section. Then problem* (P) *and equivalently problem* (Q) *attain their global solution at $s^\circ = (x^\circ, y^\circ)$.*

5. Construction of a Random Polytope

In this section, we describe a method by which the polytope P, the right-hand side $b = (b_1, b_2)^\top$, and the vectors of the upper-level objective, $c = (c_1, d_1)^\top$ are randomly generated. We have employed a method contained in [13], [12].

The matrices A and B, and the right-hand side vector b that define the polytope,

$$P = \{(x, y) |\ Ax + By \leq b,\ x \geq 0,\ y \geq 0\}$$

are generated in the following way,

Given $m = m_1 + m_2$, and $n = n_1 + n_2$, for $i = 1, \ldots, (m-1)$ and $j = 1, \ldots, n$, the elements of the matrices A, and B, and the components of the objective vector c are uniformly generated in the range $(-1, 1)$. The last row of the constraint matrix is uniformly generated in $[0, 1]$, and each right-hand side b_i, $(i = 1, \ldots, m)$ is generated in a similar way by,

$$b_i = \sum_{j=1}^{n} A_{ij} + B_{ij} + 2\mu, \quad i = 1, \ldots, m, \quad \mu \in [0, 1]$$

Obviously, we have $P \neq \phi$ and bounded.

6. Numerical Example

This section presents a numerical example generated by the procedure outlined in this paper (see also [17]).
The following matrix is randomly generated with $m = 10$ rows, and $n = 6$ columns. The number of variables controlled by the upper level is $n_1 = 4$, and the number of variables controlled by the lower level is $n_2 = 2$.

$$A = \begin{bmatrix} -6 & 1 & 1 & -3 & -9 & -7 \\ -9 & 3 & -8 & 3 & 3 & 0 \\ 4 & -10 & 3 & 5 & 8 & 8 \\ 4 & -2 & -2 & 10 & -5 & 8 \\ 9 & -9 & 4 & -3 & -1 & -9 \\ -2 & -2 & 8 & -5 & 5 & 8 \\ 0 & 4 & 5 & 10 & 0 & 0 \\ 7 & 2 & -5 & 4 & -5 & 0 \\ -9 & 9 & -9 & 5 & -5 & -4 \\ 5 & 3 & 1 & 9 & 1 & 5 \end{bmatrix}$$

$$b = (-15, -1, 25, 21, -1, 20, 26, 11, -5, 32)^{\mathsf{T}}$$

$$c_1 = (-4, 8, 1, -1)^{\mathsf{T}}, \quad d_1 = (9, -9)^{\mathsf{T}}, \quad d_2 = (-9, 9)^{\mathsf{T}}$$

The optimal solution to the problem (Q), relaxed by dropping the facially reverse convex constraint, is

$$x = (1.5000, 0, 0.8000, 0)^{\mathsf{T}}, \text{ and } \quad y = (0, 2.0750)^{\mathsf{T}}$$

with the optimal value of $\hat{z} = -23.8750$.
The procedure then selected, $s^\circ = (x^\circ, y^\circ)$ the bilevel global solution,

$$x^\circ = \begin{bmatrix} 1.575035646387827e+00 \\ 8.366801330798415e-01 \\ 1.888070342205340e-01 \\ 2.170924429657799e+00 \end{bmatrix}, \quad y^\circ = \begin{bmatrix} 1.887654467680594e+00 \\ 0 \end{bmatrix}$$

with an optimal objective value of $z^\circ = 1.540007129277554e+01$.
After the selection of s°, the algorithm systematically assigned the constraints number 1,7,9, and 10 to the lower level constraint set. The overall problem is as follows,

$$\min_{(x,y)\geq 0} \quad -4x_1 + 8x_2 + x_3 - x_4 + 9y_1 - 9y_2$$

s.t.

$$\begin{aligned}
-9x_1 + 3x_2 - 8x_3 + 3x_4 + 3y_1 &\leq -1 \\
4x_1 - 10x_2 + 3x_3 + 5x_4 + 8y_1 + 8y_2 &\leq 25 \\
4x_1 - 2x_2 - 2x_3 + 10x_4 - 5y_1 + 8y_2 &\leq 21 \\
9x_1 - 9x_2 + 4x_3 - 3x_4 - y_1 - 9y_2 &\leq -1 \\
-2x_1 - 2x_2 + 8x_3 - 5x_4 + 5y_1 + 8y_2 &\leq 20 \\
7x_1 + 2x_2 - 5x_3 + 4x_4 - 5y_1 &\leq 11
\end{aligned}$$

where y solves,

$$\min_{z\geq 0} \quad -9z_1 + 9z_2$$

s.t.

$$\begin{aligned}
-6x_1 + x_2 + x_3 - 3x_4 - 9z_1 - 7z_2 &\leq -15 \\
4x_2 + 5x_3 + 10x_4 &\leq 26 \\
-9x_1 + 9x_2 - 9x_3 + 5x_4 - 5z_1 - 4z_2 &\leq -5 \\
5x_1 + 3x_2 + x_3 + 9x_4 + z_1 + 5z_2 &\leq 32
\end{aligned}$$

7. Concluding Remarks

Test problems are an important component for the evaluation and comparison of mathematical algorithms, particularly for this class of nonconvex optimization which lacks sufficient theoretical criteria for determination of global optimality. In this paper, we presented a method for generating test problems for linear bilevel programming. The method generates test problems in which the constraints need not be separable. In addition, the method can control the selection of the optimal vertex with varying 'distance' from the optimal solution of the relaxed problem. The method requires only the solution of linear programs.

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